THE DIMENSION OF INTERSECTIONS OF CONVEX SETS

BY

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ABSTRACT

In this paper we determine the smallest number $h = h(k, n)$ having the following property: If $\mathscr{G}$ is any finite family of convex sets in Euclidean n-space, and if the intersection of every h or fewer members of $\mathscr{G}$ is at least k-dimensional, then the intersection of all members of $\mathscr{G}$ is at least k-dimensional.

Let F^n be an n-dimensional vector space over an ordered field F. Suppose $\mathscr{G}$ is a finite collection of convex subsets of F^n. A well known theorem by E. Helly [2, p. 102–108] asserts: If every subfamily $\mathscr{F}$ of $\mathscr{G}$ of cardinality $|\mathscr{F}| \leqq n+1$ has non-empty intersection, then $\cap\mathscr{G} \neq \emptyset$. Simple examples show that the number $n + 1$ appearing in Helly's theorem cannot be replaced by any smaller number. It is the purpose of this paper to determine, for $0 \leqq k \leqq n$, the smallest number $h = h(k,n)$ having the following property.

PROPERTY $A(k, n)$. *If $\mathscr{G}$ is any finite family of convex subsets of F^n, and for every subfamily $\mathscr{F}$ of $\mathscr{G}$ of cardinality $|\mathscr{F}| \leqq h$,* dim $\cap\mathscr{F} \geqq k$, *then* dim $\cap\mathscr{G} \geqq k$. ($\cap\mathscr{F}$ denotes the intersection of all members of $\mathscr{F}$, dim K denotes the dimension of the convex set K.)

Define, for $0 \leqq k \leqq n$, $0 \leqq n < \infty$:

$$c(k,n) = \begin{cases} n+1 & \text{if } k = 0, \\ \max(n+1, 2n-2k+2) & \text{if } 1 \leqq k \leqq n. \end{cases}$$

(Note that $c(k,n) = 2n - 2k + 2$ if $1 \leqq k \leqq \frac{1}{2}(n+1)$, $c(k,n) = n + 1$ if $k = 0$ or $\frac{1}{2}(n+1) \leqq k \leqq n$.)

THEOREM. $h(k,n) = c(k,n)$ *for* $0 \leqq k \leqq n$.

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REMARKS. The case $k = 0$ of our theorem is simply Helly's theorem. The case $k = n$ follows easily by applying Helly's theorem to the family $\{\text{int}\, K \colon K \in \mathcal{G}\}$. (Here $\text{int}\, K$ denotes the interior of K.) The case $k = 1$ was proved by several authors. (See [3] for a more detailed historical review.) B. Grünbaum attempted to determine $h(k,n)$ in [3], but the values given by him are incorrect for $2 < k < n-1$. These incorrect values are quoted in [2, §4.1]. (Grünbaum proves correctly that $h(k,n) \leqq 2n-k$ for $1 < k < n$, but claims without proof that the opposite inequality holds.)

The following lemma will reduce our theorem to the special case where $\mathcal{G}$ is a family of polyhedral convex cones and $\cap \mathcal{G}$ is a linear subspace of F^n.

LEMMA. *Let $\mathcal{G} = \{K_1, \cdots, K_t\}$ be a finite family of convex sets in E^n with non-empty intersection. Then there exists a family $\mathcal{G}' = \{C_1, \cdots, C_t\}$ of convex polyhedral cones with apex* 0 *having the following properties*:

1) *for any subset S of $\{1, \cdots, t\}$*, $\dim \cap \{K_i \colon i \in S\} = \dim \cap \{C_i \colon i \in S\}$;

2) $\bigcap_{i=1}^{t} C_i$ *is a linear subspace of F^n.*

PROOF. Let $T = \{1, \cdots, t\}$. For each subset S of T choose a convex polytope P_S such that $P_S \subset \cap\{K_i \colon i \in S\}$ and $\dim P_S = \dim \cap \{K_i \colon i \in S\}$. For each $i \in T$ let $Q_i = \text{conv} \cup \{P_S \colon S \subset T \text{ and } i \in S\}$. Each Q_i is a convex polytope, $Q_i \subset K_i$, and if $S \subset T$ then $P_S \subset \{\cap Q_i \colon i \in S\} \subset \cap \{K_i \colon i \in S\}$; hence, $\dim \cap \{Q_i \colon i \in S\} = \dim \cap \{K_i \colon i \in S\}$. Note that $P_T \neq \emptyset$, since $\cap \{K_i \colon i \in T\} \neq \emptyset$. Now choose a point z in the relative interior of P_T, and let $C_i = \text{cone}_0(Q_i - z) = \cup\{\lambda(Q_i - z) \colon 0 \leqq \lambda \in F\}$ be the cone with apex 0 spanned by $Q_i - z$. Each C_i is a polyhedral convex cone, and if $S \subset T$, then

$$\bigcap\{C_i \colon i \in S\} = \text{cone}_0\left(\bigcap\{Q_i \colon i \in S\} - z\right),$$

and

$$\dim \bigcap\{C_i \colon i \in S\} = \dim \bigcap\{Q_i \colon i \in S\} = \dim \bigcap\{K_i \colon i \in S\},$$

since $z \in Q_i$ for all $i \in T$. Moreover, $\bigcap_{i=1}^{t} C_i = \text{cone}_0\left(\bigcap_{i=1}^{t} Q_i - z\right)$ is a linear subspace of F^n, since $z \in \text{relint} \bigcap_{i=1} Q_i$. This concludes the proof of the lemma.

PROOF OF THEOREM. We have to show, for $1 \leqq k \leqq n$, that $c(k,n)$ is the smallest number h having property $A(k,n)$. Now, if h has property $A(k,n)$, then $h + 1$ has property $A(k,n)$ as well. It therefore suffices to show:

a) $c(k,n)$ has property $A(k,n)$.

b) $c(k,n) - 1$ does not have property $A(k,n)$.

We first prove a) by induction on n. The case $n = 1$ is trivial. Suppose $n \geqq 2$, and let $c = c(k,n)$. We have to establish, for all natural numbers t, the following assertion:

$P(t)$: *If* $\mathscr{G} = \{C_1,\cdots,C_t\}$ *is a family of* t *convex subsets of* F , *and* $\dim \cap \mathscr{F} \geqq k$ *for all* $\mathscr{F} \subset \mathscr{G}$ *with* $|\mathscr{F}| \leqq c$ *then* $\dim \cap \mathscr{G} \geqq k$.

Assertion $P(t)$ is trivial for $t \leqq c$. If $t > c$, then $P(t+1)$ follows easily from $P(t)$ as follows: If $\mathscr{G} = \{K_1,\cdots,K_{t+1}\}$, and $\dim \mathscr{F} \geqq k$ for all $\mathscr{F} \subset \mathscr{G}$ with $|\mathscr{F}| \leqq c$, then $\dim \cap \mathscr{F} \geqq k$ for all $\mathscr{F} \subset \mathscr{G}$ with $|\mathscr{F}| \leqq t$, by $P(t)$. Let $\mathscr{G}' = \{K_i \cap K_{t+1}: 1 \leqq i \leqq t\}$. Then $\dim \cap \mathscr{F}' \geqq k$ for all $\mathscr{F}' \subset \mathscr{G}'$ with $|\mathscr{F}'| = t-1 \geqq c$, and therefore $\dim \cap \mathscr{G}' \geqq c$, again by $P(t)$. But $\cap \mathscr{G}' = \cap \mathscr{G}$.

It remains to prove assertion $P(c+1)$. Let $\mathscr{G} = \{C_1,\cdots,C_{c+1}\}$ be a family of $c+1$ convex subsets of F^n, and suppose $\dim \cap \mathscr{F} \geqq k$ for all $\mathscr{F} \subset \mathscr{G}$ with $|\mathscr{F}| \leqq c$. We shall show since that $\dim \cap \mathscr{G} \geqq k$. By Helly's theorem we know that $\cap \mathscr{G} \neq \emptyset$, since $c \geqq n+1$. We may therefore assume, using the lemma, that each C_i is a polyhedral convex cone with apex 0, and that $\cap \mathscr{G}$ is a linear subspace of F^n.

Suppose $\dim \cap \mathscr{G} = l < k$. We distinguish two cases and obtain a contradiction in each case.

Case 1. $l > 0$.

Let $H \subset F^n$ be a hyperplane through 0, such that $\cap \mathscr{G} \not\subset H$. Hence $\dim H \cap (\cap \mathscr{G}) = l-1 < k-1$. If $\mathscr{F} \subset \mathscr{G}$, then H is not a supporting hyperplane of $\cap \mathscr{F}$, since H does not even support $\cap \mathscr{G}$. It follows that if $\mathscr{F} \subset \mathscr{G}$, $|\mathscr{F}| \leqq c$, then $\dim H \cap (\cap \mathscr{F}) = \dim \cap \mathscr{F} - 1 \geqq k-1$. Define $\mathscr{G}' = \{H \cap C_1,\cdots,H \cap C_{c+1}\}$ Note that $c(k-1,n-1) \leqq c(k,n) = c$. Therefore, if $\mathscr{F} \subset \mathscr{G}'$ and $|\mathscr{F}| \leqq c(k-1,n-1)$, then $\dim \mathscr{F} \geqq k-1$. It follows by the induction hypothesis that $\dim \cap \mathscr{G}' \geqq k-1$. But $\cap \mathscr{G}' = H \cap (\cap \mathscr{G})$, and we have seen that $\dim H \cap (\cap \mathscr{G})) = l-1 < k-1$, a contradiction.

Case 2. $l = 0$.

In this case $\cap \mathscr{G} = \bigcap_{i=1}^{c+1} C_i = \{0\}$. Define, for $1 \leqq i \leqq c+1$: $C_{-i} = \cap \{C_j: 1 \leqq j \leqq c+1, j \neq i\}$. By assumption, $\dim C_{-i} \geqq k$ for $1 \leqq i \leqq c+1$.

Among all $(c+1)$-tuples $(a_1,\cdots,a_{c+1})$ such that $a_i \in C_{-i}$ for $1 \leqq i \leqq c+1$ choose one, say $(b_1,\cdots,b_{c+1})$, of maximal rank (linear dimension). Suppose $\operatorname{rank}(b_1,\cdots,b_{c+1}) = m$ $(m \leqq n)$ and assume, for convenience's sake, that $\operatorname{rank}(b_1,\cdots,b_m) = m$. Let $B = \{b_1,\cdots,b_m\}$. If $m < i \leqq c+1$, and $a_i \in C_{-i}$, than a_i is a linear combination of B, because of the maximal rank of B. Suppose

$a_i = \sum_{r\in S_+}\alpha_r b_r - \sum_{r\in S_-}\alpha_r b_r$, where S_+, S_- are disjoint (possibly empty) subsets of $\{1,\cdots,m\}$, and $0<\alpha_r\in F$ for $r\in S_+\cup S_-$. Let $z = \sum_{r\in S_+}\alpha_r b_r = a_i + \sum_{r\in S_-}\alpha_r b_r$. If $1\leqq j\leqq c+1$ and $j\notin S_+$, then $b_r\in C_{-r}\subset C_j$ for all $r\in S_+$. Hence $z\in C_j$. If $j\in S_+$, then $j\notin S_-\cup\{i\}$, and again $b_r\in C_{-r}\subset C_j$ for all $r\in S_-$ and also $a_i\in C_{-i}\subset C_j$. Hence $z\in C_j$.

It follows that $z\in\cap\mathscr{G} = \{0\}$, and therefore $S_+ = \emptyset$.

If $m+1\leqq i<j\leqq c+1$, and $a_i\in C_{-i}$, $a_j\in C_{-j}$, then we have, by the preceding paragraph.:

$$a_i = -\sum_{r\in S_i}\alpha_r b_r, \qquad a_j = -\sum_{r\in S_j}\alpha'_r b_r$$

where $S_i, S_j\subset\{1,\cdots,m\}$ and $\alpha_r,\alpha'_r>0$.

Let

$$z = -\sum_{r\in S_i\cap S_j}\min(\alpha_r,\alpha'_r)b_r,$$

$$T = (S_i - S_j)\cup\{r\in S_i\cap S_j : \alpha_r>\alpha'_r\},$$

$$T' = (S_j - S_i)\cup\{r\in S_i\cap S_j : \alpha'_r>\alpha_r\}.$$

T and T' are disjoint subsets of B.

Since

$$z = a_i + \sum_{r\in S_i}\alpha_r b_r - \sum_{r\in S_i\cap S_j}\min(\alpha_r,\alpha'_r)b_r$$

$$= a_j + \sum_{r\in S_j}\alpha'_r b_r - \sum_{r\in S_i\cap S_j}\min(\alpha_r,\alpha'_r)b_r,$$

we see that z can be written as a positive linear combination of either one of the disjoint sets $\{a_i\}\cup\{b_r : r\in T\}$, $\{a_j\}\cup\{b_r : r\in T'\}$. It follows as in the preceding paragraph, that $z\in C_r$ for all $1\leqq r\leqq c+1$. Hence $z=0$, $S_i\cap S_j=\emptyset$.

Now choose points $a_i\in C_{-i}$ for $m+1\leqq i\leqq c+1$. With each point a_i we associate a subset S_i of $\{1,\cdots,m\}$, as above, suuch that $\alpha_i = -\sum_{r\in S_i}\alpha_r b_r$, $0<\alpha_r\in F$.

Since $\dim C_{-i}\geqq k$, we can choose a point $a_i\in C_{-i}$ which is not a linear combination of fewer than k points in B, and therefore $|S_i|\geqq k$. The sets $S_{m+1},\cdots,S_{c+1}$ are pairwise disjoint. Hence

$$(c+1-m)k\leqq m.$$

Since $m\leqq n$ we obtain $(c+1-n)k\leqq n$, or $c(k,n)=c\leqq n/k+n-1$. To derive a contradiction, we show that $c(k,n) > n/k + n - 1$. Indeed, if $\frac{1}{2}(n+1) \leqq k \leqq n$, then $c(k,n) = n + 1 > n/k + n - 1$. If $1 \leqq k < \frac{1}{2}(n+1)$ then $c(k,n) = 2n - 2k + 2$ and $n \geqq 2k$, and therefore

$$(c(k,n) + 1 - n)k - n = (n + 3 - 2k)k - n = n(k-1) + 3k - 2k^2$$
$$\geqq 2k(k-1) + 3k - 2k^2 = k > 0.$$

This concludes the proof of a).

We now prove b), i.e., $c(k,n) - 1$ does not have property $A(k,n)$. This is done by constructing examples of finite families $\mathscr{G}$ of convex sets in F^n, such that $\dim \cap \mathscr{G} < k$, but $\dim \cap \mathscr{F} \geqq k$ for every subfamily $\mathscr{F}$ of $\mathscr{G}$ with $|\mathscr{F}| \leqq c(k,n) - 1$.

Let $e_1, \cdots, e_n$ be a linear basis of F^n. Each point $x \in F^n$ can be expressed uniquely as $x = \sum_{i=1}^{n} \xi_i e_i$, $\xi_i \in F$.

Case 1. $1 \leqq k \leqq \frac{1}{2}(n+1)$. In this case $c(k,n) = 2n - 2k + 2$. Let $\mathscr{G} = \{A_1, \cdots, A_{2n}\}$, where

$$A_i = \{x \in F^n : \xi_i \geqq 0\},$$
$$A_{n+i} = \{x \in F^n : \xi_i \leqq 0\} \quad \text{for} \quad 1 \leqq i \leqq n.$$

Then $\cap \mathscr{G} = \{0\}$, i.e., $\dim \cap \mathscr{G} = 0$, but the intersection of every $2n - 2k + 1$ members of $\mathscr{G}$ is at least k-dimensional.

Case 2. $k = 0$ or $\frac{1}{2}(n+1) \leqq k \leqq n$. In this case $c(k,n) = n + 1$. Let $\mathscr{G}' = \{A'_0, A'_1, \cdots, A'_n\}$ where $A'_0 = \{x \in F^n : \sum_{i=1}^{n} \xi_i \leqq -1\}$, and $A'_i = \{x \in F^n : \xi_i \geqq 0\}$ for $1 \leqq i \leqq n$. Then $\cap \mathscr{G}' = \emptyset$. I.e., $\dim \cap \mathscr{G}' = -1$, but the intersection of every n members of $\mathscr{G}'$ is n-dimensional.

This concludes the proof of b), and we have therefore proved our theorem.

We now outline an alternate proof of our theorem, which uses the following result of Bonnice and Klee [1, corollary 2.12]:

Let e be an integer, $0 \leqq e \leqq n$. If $\mathscr{G}$ is a finite family of polyhedral convex cones with apex 0 *in F^n, and if $\cap \mathscr{G} = \{0\}$*, then *there exists a subfamily $\mathscr{F}$ of $\mathscr{G}$ with $|\mathscr{F}| \leqq \max(n+1, 2(n-e))$ and* $\dim \cap \mathscr{F} \leqq e$.

(The original version of corollary 2.12 in [1] deals with an arbitrary (possibly infinite) family of closed convex cones in R^n. The restriction to finite families of polyhedral cones is necessary if we work with an arbitrary ordered field F instead of the real field R).

Let $\mathscr{G} = \{C_1, \cdots, C_t\}$ be a finite family of convex polyhedral cones with apex 0 in F^n, and suppose $\cap \mathscr{G}$ is a linear subspace of F^n. Assume that $\dim \cap \mathscr{F} \geqq k$ for all $\cap \mathscr{F} \subset \mathscr{G}$ with $|\mathscr{F}| \leqq c(k,n)$ but $\dim \cap \mathscr{G} = l < k$ $(1 \leqq k \leqq n)$.

If $l > 0$, then we reduce the dimension by one and obtain a contradiction to the case $k-1$, $n-1$ of our theorem, as in case 1 of the previous proof. If $l = 0$, then we apply the quoted result of Bonnice and Klee with $e = k-1$, and find a subfamily $\mathcal{F}$ of $\mathcal{G}$ with $|\mathcal{F}| \leqq \max(n+1, 2(n-k+1)) = c(k,n)$ and $\dim \cap \mathcal{F} \leqq k-1 < k$, a contradiction.

REMARK. The result quoted above, which is essentially the bracketed part of [1, 2.12] follows clearly from our theorem, with $k = e+1$. The unbracketed part of [1,2.12] follows quite easily from the bracketed part. The principal result of section 2 of [1], theorem 2.5, follows easily from [1, 2.12] via [1, 2.11] and [1, 2.6]. The pıoof given in this paper can thus serve as an alternative approach to section 2 of [1].

We conclude the paper with an open problem due to Micha A. Perles. Let $\mathcal{G} = \{K_1, \cdots, K_t\}$ be a finite family of convex subsets of F^n. Let $T = \{1, \cdots, t\}$ and define for $S \subset T$: $d(\mathcal{G}, S) = \dim \cap \{K_i\colon i \in S\}$. From our theorem it follow that

$$d(\mathcal{G}, T) = \min\{d(\mathcal{G}, S)\colon S \subset T, |S| \leqq 2n\}.$$

The value of $d(\mathcal{G}, T)$ is thus determined by the values of $d(\mathcal{G}, S)$ for $S \subset T$, $|S| \leqq 2n$. It is not hard to show that the number $2n$ in the previous statement can be replaced by $2n-1$ if $n \geqq 2$. Can it be replaced by any smaller number? More specifically, we ask for the smallest number $h = h(n)$ having the following property:

If $\mathcal{G} = \{K_i\colon i \in T\}$ and $\mathcal{G}' = \{K'_i\colon i \in T\}$ are two finite families of convex subsets of F^n, and if $d(\mathcal{G}, S) = d(\mathcal{G}', S)$ for all $S \subset T$ with $|S| \leqq h(n)$ then $d(\mathcal{G}, T) = d(\mathcal{G}', T)$. All we know is that $n+1 \leqq h(n) \leqq 2n-1$ for $n \geqq 2$. I conjecture that $h(n) = n+1$ for all n.

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Note added in proof: I have now proved that $h(n) = n+1$ for all n.

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